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CHOQUET THEORY FOR SIGNED MEASURES

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Abstract. We introduce the notion of barycenter for a class of non-necessarily positive Radon measures and prove on this basis several inequalities which extend classical results such as Steffensen's inequality, Fink's version of the Hermite-Hadamard inequality, Fuchs 'extension of the majorization inequality of Hardy-Littlewood-Polýa etc.

A classical result in Real Analysis is the *Hermite-Hadamard inequality*, which gives us an estimate of the mean value of a continuous convex function. Precisely, if $f : [a, b] \to \mathbf{R}$ is such a function, then

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b f(x)\,dx \leqslant \frac{f(a)+f(b)}{2}. \tag{HH}$$

See D. S. Mitrinović and I. B. Lacković [8] for a complete history of this result.

In a recent paper, A. M. Fink [3] noticed that a Hermite-Hadamard type inequality is still available for certain real Radon measures. His extension of (HH) is as follows:

$$f(x_\mu) \leqslant \frac{1}{\mu([a,b])}\int_a^b f(x)\,d\mu(x) \leqslant \frac{b-x_\mu}{b-a}\cdot f(a) + \frac{x_\mu - a}{b-a}\cdot f(b) \tag{FHH}$$

for every continuous convex function $f : [a, b] \to \mathbf{R}$ *and every real Radon measure* μ *on* $[a, b]$, *provided that* μ *is "end positive" in the sense that*

$$\mu([a,b]) > 0,\ \int_a^t (t-x)\,d\mu(x) \geqslant 0 \text{ and } \int_t^b (x-t)\,d\mu(x) \geqslant 0 \tag{EP}$$

for every $t \in [a, b]$. Here $x_\mu = \int_a^b x\,d\mu(x)/\mu([a,b])$ represents the barycenter of μ.

As we noticed in [9], in the case of Radon probability measures, Fink's result can be easily deduced from the Choquet theory, a theory whose highlights have been presented by R. R. Phelps in his book [11]. The purpose of the present paper is to indicate a slight generalization of the Choquet theory for signed measures, so that the entire result of Fink be covered. It turns out that this extension encompasses many other important results such as Steffensen's inequality and the majorization principle of Hardy, Littlewood and Polýa (as extended by L. Fuchs [4]).

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1. The barycenter of an essentially positive measure

Throughout this paper K will denote a compact convex subset of a locally convex Hausdorff space E and $C(K,\mathbf{R})$ will denote the space of all continuous real functions on K. The classical Choquet theory relates the geometry of K with the cone $C,$ of all continuous convex real functions on K; $C-C$ is dense in $C(K,\mathbf{R})$, by the Stone-Weierstrass theorem. Much of that theory makes use of the space $A(K,\mathbf{R})=C\cap -C,$ of all continuous affine functions on K. This is a rich space, as it contains

$$E'|_K+\mathbf{R}\cdot 1=\{x'|_K+\alpha;\ x'\in E' \text{ and } \alpha\in\mathbf{R}\}$$

as a dense subspace. See [11], Ch. 4, Proposition 4.5.

An easy consequence of the Hahn-Banach separation theorem is that the convex functions can be described as envelopes of affine functions. In fact, the following assertion holds:

LEMMA 1. *For every* $f\in C$ *there exists a sequence of affine functions* $h_n\in A(K,\mathbf{R})$ *such that* $f=\sup h_n$.

Proof. See [11], page 19, where the case of concave functions is described. ■

The connection between the points of K and the positive functionals on $C(K,\mathbf{R})$ makes the object of Choquet's theory, as presented in [11]. The key notion is that of barycenter. Every point of K can be seen as the barycenter of a Radon probability measure on $K,$ and every such a measure has a barycenter. We shall enlarge this picture, by allowing the participation of certain signed measures:

DEFINITION 1. A *Popoviciu measure* (abbreviated, a $P-$measure) is any real Radon measure μ on K such that

$$\mu(K)>0 \quad\text{and}\quad \int_K f^+(x)\,d\mu(x)\geqslant 0 \quad\text{for every}\quad f\in C. \tag{PM}$$

When K is an interval $[a,b]$ and μ is a real Radon measure on $[a,b]$ with $\mu([a,b])>0,$ the condition (PM) coincides with the condition of end positivity (EP) mentioned above, a fact which was known to T. Popoviciu [12]. In fact, (PM) yields

$$\mu(K)>0 \quad \textit{and} \quad \int_K (x'(x)+t)^+\ d\mu(x)\geqslant 0 \quad \text{for every } x'\in E' \text{ and every } t\in\mathbf{R} \tag{wPM}$$

and the dual of $\mathbf{R}$ consists only of homoteties $x':x\to sx$. T. Popoviciu's argument for the other implication, (EP) $\Rightarrow$ (PM), was as follows: If $f\geqslant 0$ is a piecewise linear continuous and convex function, then f can be represented as a finite combination with non-negative coefficients of functions of the form $1,\ (x-t)^+$ and $(t-x)^+,$ so that

$$\int_K f(x)\,d\mu(x)\geqslant 0;$$

in the general case, approximate f^+ by piecewise linear continuous and convex functions. It is worth noticing that T. Popoviciu [12] was interested in a slightly different problem, precisely, when a real Radon measure on an interval $[a, b]$ is non-negative for all $n-$ convex functions on that interval.

An alternative argument for $(\mathrm{EP}) \Rightarrow (\mathrm{PM})$, based on the integral representation of convex functions on intervals, was done by Fink [3], Theorem 1.

EXAMPLE 1. *(The discrete case).* Suppose that $x_1 \leqslant \ldots \leqslant x_n$ are real points and $p_1, \ldots, p_n$ are real weights. According to the discussion above, the discrete measure $\mu = \sum_{k=1}^{n} p_k \, \delta_{x_k}$ is a Popoviciu measure if and only if

$$\sum_{k=1}^{n} p_k > 0, \quad \sum_{k=1}^{m} p_k(x_m - x_k) \geqslant 0 \quad \text{and} \quad \sum_{k=m}^{n} p_k(x_k - x_m) \geqslant 0 \tag{dEP}$$

for every $m \in \{1, \ldots, n\}$. A special case when (dEP) holds is the following, used by Steffensen in his famous extension of Jensen's inequality:

$$\sum_{k=1}^{n} p_k > 0, \quad \text{and} \quad 0 \leqslant \sum_{k=1}^{m} p_k \leqslant \sum_{k=1}^{n} p_k, \quad \text{for every } m \in \{1, \ldots, n\}. \tag{dSt}$$

EXAMPLE 2. (*The continuous case*). In the case of absolutely continuous measures $d\mu = p(x)\, dx$, the condition (EP) reads as:

$$\int_a^b p(x)\, dx > 0, \quad \int_a^t (t - x)p(x)\, dx \geqslant 0 \quad \text{and} \quad \int_t^b (x - t)p(x)\, dx \geqslant 0 \tag{cEP}$$

for every $t \in [a, b]$. As a particular case, we obtain that $(x^2 + a)\, dx$ is a Popoviciu measure on $[-1, 1]$ if $a > -1/3$ (though non-positive if $a \in (-1/3, 0)$).

A stronger (but more suitable) condition than (cEP) is the following:

$$\int_a^b p(x)\, dx > 0 \quad \text{and} \quad 0 \leqslant \int_a^t p(x)\, dx \leqslant \int_a^b p(x)\, dx \text{ for every } t \in [a, b]. \tag{cSt}$$

Integrating inequalities is not generally possible in the framework of signed measures. However, for the Popoviciu measures this is possible under certain restrictions, as (PM) yields easily the following implication:

LEMMA 2. *Suppose that μ is a Popoviciu measure on K. Then $h \in A(K, \mathbf{R})$, $f \in C$ and $h \leqslant f$ implies*

$$\int_K h(x)\, d\mu(x) \leqslant \int_K f(x)\, d\mu(x).$$

An immediate consequence is as follows:

COROLLARY 1. *Suppose that μ is a Popoviciu measure on K and f is an affine function on K such that $\alpha \leqslant f \leqslant \beta$ for some real numbers α, β. Then*

$$\alpha \leqslant \frac{1}{\mu(K)} \int_K f(x)\, d\mu(x) \leqslant \beta.$$

As a consequence of the above corollary, if μ is a Popoviciu measure on $K,$ then $\|\mu|A(K,\mathbf{R})\| = \mu(K)$. However, the norm of $\mu/\mu(K)$ as a Radon measure on K (i.e., as a functional on $C(K,\mathbf{R})$) can be arbitrarily large. In fact,

$$\int_{-1}^{1}(x^2+a)dx = \frac{2}{3}+2a$$

and

$$\left(\frac{2}{3}+2a\right)^{-1}\int_{-1}^{1}\left|x^2+a\right| dx = \frac{1}{1+3a}$$

for $a > -1/3$. This makes a serious difference with respect to the case of positive Radon measures, where the norm of $\mu/\mu(K)$ is 1.

LEMMA 3. *Every Popoviciu measure μ on K admits a barycenter i.e., a point x_μ in K such that*

$$f(x_\mu) = \frac{1}{\mu(K)}\int_K f(x)\,d\mu(x) \tag{B}$$

for every continuous linear functional f on E.

The barycenter x_μ is unique with this property. This is a consequence of the separability of the topology of the ambient space E.

Proof. We have to prove that

$$\left(\bigcap_{f\in E'} H_f\right)\cap K \neq \emptyset$$

where H_f denotes the closed hyperplane $\{x;\ f(x)=\mu(f)/\mu(K)\}$ associated to $f \in E'$. As K is compact, it suffices to prove that

$$\left(\bigcap_{k=1'}^{n} H_{f_k}\right)\cap K \neq \emptyset$$

for every finite family $f_1,\ldots,f_n$ of functionals in E'. Equivalently, attaching to such a family of functionals the operator

$$T : K \to \mathbf{R}^n, \quad T(x) = (f_1(x),\ldots,f_n(x))$$

we have to prove that $T(K)$ contains the point $p = \frac{1}{\mu(K)}(\mu(f_1),\ldots,\mu(f_n))$. For, if $p \notin T(X),$ then a separation argument yields an $a=(a_1,\ldots,a_n)\in\mathbf{R}^n$ such that

$$\langle p, a\rangle > \sup_{x\in K}\langle T(x), a\rangle$$

i.e.,

$$\frac{1}{\mu(K)}\sum_{k=1}^{n} a_k\mu(f_k) > \sup_{x\in K}\sum_{k=1}^{n} a_k f_k(x).$$

Then $g=\sum_{k=1}^{n} a_k f_k$ will provide an example of a continuous affine function on K for which $\mu(g) > \sup_{x\in K} g(x),$ a fact which contradicts Lemma 2. ■

When E is the Euclidean $n-$ dimensional space, the norm and the weak convergence are the same, so that the barycenter of every Popoviciu measure μ on $K \subset \mathbf{R}^n$ is given by the formula

$$x_\mu = \frac{1}{\mu(K)} \int_K x \, d\mu(x).$$

Two Popoviciu measures μ and ν on $K,$ are said to be *equivalent* (abbreviated, $\mu \sim \nu$) provided that

$$\int_K f(x) \, d\mu(x) = \int_K f(x) \, d\nu(x) \quad \text{for every } f \in A(K, \mathbf{R}).$$

Using the denseness of $E'|_K + \mathbf{R} \cdot 1$ into $A(K, \mathbf{R}),$ we can rewrite the fact that x is the barycenter of μ as

$$\mu \sim \delta_x.$$

The following result extends the left side part of the Hermite-Hadamard inequality:

THEOREM 1. (*The generalized Jensen-Steffensen inequality*). *Suppose that μ is a real Radon measure on K with $\mu(K) > 0$. Then*

$$f(x_\mu) \leqslant \frac{1}{\mu(K)} \int_K f(x) \, d\mu(x) \text{ for every continuous convex function } f \text{ on } K$$

if (*and only if*) μ *is a Popoviciu measure.*

Proof. The Necessity is clear. The Sufficiency follows from Lemmas 1 and 3 which give us

$$\begin{aligned} f(x_\mu) &= \sup \{ h(x_\mu); \; h \in A(K, \mathbf{R}), \; h \leqslant f \} \\ &= \sup \left\{ \frac{1}{\mu(K)} \int_K h \, d\mu; \; h \in A(K, \mathbf{R}), \; h \leqslant f \right\} \\ &\leqslant \frac{1}{\mu(K)} \int_K f \, d\mu. \; \blacksquare \end{aligned}$$

We shall illustrate Theorem 1 by an application to Steffensen's inequality (in the discrete case).

Suppose that $x_1 \leqslant \ldots \leqslant x_n$ are real points in an interval I and $p_1, \ldots, p_n$ are real weights such that the condition (dSt) above is verified; considering the partial sums $S_k = \sum_{i=1}^k p_i,$ this means

$$0 \leqslant S_k \leqslant S_n \quad \text{and} \quad S_n > 0.$$

Then the discrete measure $\mu = \sum_{k=1}^n p_k \, \delta_{x_k}$ is a Popoviciu measure with barycenter

$$x_\mu = \frac{1}{S_n} \sum_{k=1}^n p_k \, x_k.$$

According to Theorem 1 above, we are led to the classical Steffensen inequality $([7]$, p. $109)$: *For every continuous convex function* $f : I \to \mathbf{R}$,

$$f\left(\frac{1}{S_n}\sum_{k=1}^{n} p_k x_k\right) \leqslant \frac{1}{S_n}\sum_{k=1}^{n} p_k f(x_k).$$

The continuous case can be obtained in a similar way. It is worth noticing that Steffensen's inequality also holds under the more general condition (dEP).

Another straightforward application of Theorem 1 is the following inequality of G. Szegö: *If* $a_1 > a_2 > .. > a_{2m-1} > 0$ *and* f *is a convex function in* $[0, a_1]$, *then*

$$\sum_{k=1}^{2m-1} (-1)^{k-1} f(a_k) \geqslant f\left(\sum_{k=1}^{2m-1} (-1)^{k-1} a_k\right).$$

This corresponds to the measure $\mu = \sum_{k=1}^{2m-1} (-1)^{k-1} \delta_{a_k}$, whose barycenter is $x_\mu = \sum_{k=1}^{2m-1} (-1)^{k-1} a_k$.

The reader can verify easily that many other inequalities with alternating signs are consequences of Theorem 1.

2. The case of $0-$ mass measures

The discussion above left open the case of real Radon measures μ on K with $\mu(K) = 0$. The analogue of Theorem 1 is the following result:

PROPOSITION 1. *Let* μ *be a real Radon measure on* K *such that* $\mu(K) = 0$ *and*

$$\int_K f^+ \, d\mu(x) \geqslant 0 \quad \text{for every continuous convex function } f \text{ on } K.$$

Then

$$\int_K f(x)\, d\mu(x) \geqslant 0 \quad \text{for every continuous convex function } f \text{ on } K.$$

Proof. In fact, by replacing μ by $\mu_\varepsilon = \mu + \varepsilon\delta_z$ (where z is any point of K and $\varepsilon > 0$) we obtain a Popoviciu measure, which makes possible to apply Theorem 1. Then

$$f(x_\mu) \cdot (\mu(K) + \varepsilon) \leqslant \int_K f(x)\, d\mu(x) + \varepsilon f(z)$$

for every continuous convex function f on K, and the conclusion follows by letting $\varepsilon \to 0$. ■

On intervals we can prove a better result:

THEOREM 2. *Let μ be a real Radon measure on $[a,b]$ such that $\mu([a,b]) = 0$ and*

$$\int_a^t (t-x)\,d\mu(x) \geqslant 0 \quad \text{and} \quad \int_t^b (x-t)\,d\mu(x) \geqslant 0$$

for every $t \in \mathbf{R}$. Then

$$\int_a^b f(x)\,d\mu(x) \geqslant 0$$

for every convex function f on $[a,b]$.

Proof. See the Popoviciu approximation of convex functions, noticed in the preceding section. ■

As an immediate consequence we obtain the following extension of the majorization principle:

THEOREM 3. (L. Fuchs [4]; see also [7], pp. 165-166). *Let $f : [a,b] \to \mathbf{R}$ be a convex function. Then for every $x_1, \ldots, x_n, y_1, \ldots, y_n \in [a,b]$ and every $p_1, \ldots, p_n \in \mathbf{R}$ such that*

i) $x_1 > \ldots > x_n, \quad y_1 > \ldots > y_n$

ii) $\sum_{k=1}^{r} p_k x_k \leqslant \sum_{k=1}^{r} p_k y_k$ *for every* $r = 1, \ldots, n-1$

iii) $\sum_{k=1}^{n} p_k x_k = \sum_{k=1}^{n} p_k y_k$

we have the inequality

$$\sum_{k=1}^{n} p_k f(x_k) \leqslant \sum_{k=1}^{n} p_k f(y_k).$$

Proof. (In the case when all weights are non-negative). It suffices to verify that the measure $\mu = \sum_{k=1}^{n} p_k (\delta_{y_k} - \delta_{x_k})$ fulfils the hypotheses of Theorem 2 above. For example, to check that

$$\int_a^t (t-x)\,d\mu(x) = \sum_{k=1}^{n} p_k (t-y_k)^+ - \sum_{k=1}^{n} p_k (t-x_k)^+ \geqslant 0$$

for all t it suffices to restrict to the case when $t = x_r$. Or, in this case

$$\begin{aligned}\sum_{k=1}^{n} p_k (t-y_k)^+ - \sum_{k=1}^{n} p_k (t-x_k)^+ &= \sum_{k=1}^{n} p_k (x_r-y_k)^+ - \sum_{k=1}^{n} p_k (x_r-x_k)^+ \\ &\geqslant \sum_{k=r+1}^{n} p_k (x_r-y_k) - \sum_{k=r+1}^{n} p_k (x_r-x_k) \\ &= \sum_{k=r+1}^{n} p_k (x_k-y_k) \geqslant 0. \ ■\end{aligned}$$

We pass now to the case of absolutely continuous measures:

PROPOSITION 2. *Let $p(x)$ be a $[a,b]$ a continuous or a monotonic density $p(x)$ on an interval $[a,b]$, such that*

$$\int_a^b p(x)\,dx = 0 \quad \text{and} \quad \int_a^t p(x)\,dx \geqslant 0, \quad \int_t^b p(x)\,dx \geqslant 0$$

for every $t \in [a,b]$. Then

$$\int_a^b f(x)\,d\mu(x) \geqslant 0$$

for every convex function f on $[a,b]$.

This allows us to retrieve the following remark due to L. Lupaş [6]: *Suppose that $g : [-a,a] \to R$ is an even function, nondecreasing on $[0,a]$ and $f : [-a,a] \to R$ is a convex function. Then*

$$\frac{1}{2a}\int_{-a}^{a} f(x)g(x)\,dx \geqslant \left(\frac{1}{2a}\int_{-a}^{a} f(x)dx\right)\left(\frac{1}{2a}\int_{-a}^{a} g(x)\,dx\right).$$

In fact, $p(x) = g(x) - \frac{1}{2a}\int_{-a}^{a} g(x)\,dx$ fulfils the conditions of Proposition 2 above.

3. The extension of Choquet's Theorem

The extension of the right hand inequality in (HH) is a bit more subtle and makes the object of Choquet's theory, briefly summarized in the sequel. Given two Popoviciu measures μ and λ on K, we say that μ is *majorized* by λ (i.e., $\mu \prec \lambda$) if

$$\frac{1}{\mu(K)}\int_K f(x)\,d\mu(x) \leqslant \frac{1}{\lambda(K)}\int_K f(x)\,d\lambda(x)$$

for every continuous convex function $f : K \to \mathbf{R}$. The relation $\prec$ is a partial ordering on the set of all essentially positive Radon measures on K; use the denseness of $C - C$ in $C(K,\mathbf{R})$.

Notice that $\mu \sim \delta_x$ implies $\delta_x \prec \mu$ (by Theorem 1, the generalized Jensen-Steffensen inequality).

THEOREM 4. (The generalization of Choquet's Theorem). *Let μ be a Popoviciu measure on a metrizable compact convex subset K of a locally convex Hausdorff space E. Then there exists a probability Radon measure λ on K such that the following two conditions are verified:*

i) $\lambda \succ \mu$ and λ and μ have the same barycenter;

ii) The set Ext K of all extremal points of K is a $G_\delta-$subset of K and λ is concentrated on Ext K (i.e., $\lambda(K \setminus \operatorname{Ext} K) = 0$).

Under the hypotheses of Theorem 4 we get

$$f(x_\mu) \leqslant \frac{1}{\mu(K)}\int_K f(x)\,d\mu(x) \leqslant \int_{\operatorname{Ext} K} f(x)\,d\lambda(x) \tag{Ch}$$

for every continuous convex function $f : K \to \mathbf{R}$, a fact which represents a full extension of (HH) in the case of *metrizable* compact convex sets. Notice that the right part of (Ch) reflects the *maximum principle* for convex functions.

In general, λ is not unique, except for the case of simplices; see [11], ch. 9.

Proof. (of Theorem 4). The fact that the set $Ext\ K$ of all extremal points of K is a $G_\delta-$subset constitutes Proposition 1.3 in [11]. Here the assumption of metrizability is essential. We pass now to the existence of λ.

The upper envelope of a function f in $C(K, \mathbf{R})$,

$$\overline{f}(x) = \inf \{h(x);\ h \in A(K, \mathbf{R}) \text{ and } h \geqslant f\}$$

is concave, bounded and upper semicontinuous. Moreover:

i) $f \leqslant \overline{f}$ and $f = \overline{f}$ if f is concave.

ii) If $f, g \in C(K, \mathbf{R})$, then $\overline{f+g} \leqslant \overline{f} + \overline{g}$.

See [11], p. 19, for details. These properties show that the functional

$$p : C(K, \mathbf{R}) \to \mathbf{R}, \quad p(f) = \mu(\overline{f})/\mu(K)$$

is subadditive and positive-homogeneous. According to the generalized Jensen-Steffensen inequality, p dominates the linear functional

$$L : A(K, \mathbf{R}) \to \mathbf{R}, \quad L(h) = h(x_\mu).$$

By the Hahn-Banach extension theorem, there exists a functional $\nu : C(K, \mathbf{R}) \to \mathbf{R}$ which extends L and

$$\nu(f) \leqslant p(f) \text{ for every } f \in C(K, \mathbf{R}).$$

If $f \in C(K, \mathbf{R})$, with $f \leqslant 0$, then $\overline{f} \leqslant 0$ and $\mu(\overline{f}) \leqslant 0$ (as μ is a Popoviciu measure). This fact shows that $\nu(f) \leqslant 0$, i.e., ν is a positive Radon measure. Since $\nu(1) = L(1) = 1$, ν is actually a Radon probability measure.

On the other hand, if $f \in C$ then $\nu(-f) \leqslant \mu(\overline{-f})/\mu(K) = \mu(-f)/\mu(K)$, which yields $\mu \prec \nu$. Moreover, μ and ν have the same barycenter (as they agree on $A(K, \mathbf{R})$.The proof ends by choosing a maximal Radon probability measure $\lambda \succ \nu$, which does the job in the classical case of Choquet theory. The existence of λ is motivated in [11], ch. 4. ■

According to the above discussion, if $K = [a, b]$, then necessarily λ is a convex combination of the Dirac measures ε_a and ε_b, say $\lambda = (1-\alpha)\varepsilon_a + \alpha\varepsilon_b$. This remark yields Fink's Hermite-Hadamard type inequality [3]:

$$\frac{1}{\mu([a, b])} \int_a^b f(x)\, d\mu(x) \leqslant \frac{b - x_\mu}{b - a} \cdot f(a) + \frac{x_\mu - a}{b - a} \cdot f(b) \tag{FHH}$$

for every continuous convex functions $f : [a, b] \to \mathbf{R}$ *and every Popoviciu measure* μ *on* $[a, b]$; as usually, x_μ denotes the barycenter of μ, i.e,

$$x_\mu = \frac{1}{\mu([a, b])} \int_a^b x\, d\mu(x).$$

In fact, checking

$$\frac{1}{\mu([a,b])}\int_a^b f(x)\,d\mu(x) \leqslant (1-\alpha)\cdot f(a) + \alpha\cdot f(b)$$

for $f(x) = (x-a)/(b-a)$ and $f(x) = (b-x)/(b-a)$ we obtain

$$\alpha \geqslant \frac{x_\mu - a}{b-a} \quad \text{and respectively } 1-\alpha \geqslant \frac{b-x_\mu}{b-a}$$

i.e., $\alpha = (x_\mu - a)/(b-a)$.

The argument above can be extended easily for all continuous convex functions defined on n-dimensional simplices $K = [A_0, A_1, \ldots, A_n]$ in $\mathbf{R}^n$. Then the corresponding analogue of (F) for Popoviciu measures μ *on* K will read as

$$f(X_\mu) \leqslant \frac{1}{\mu([a,b])}\int_K f(x)\,d\mu \leqslant \sum_{k=0}^{n} \mathrm{Vol}_n\left([A_0, A_1, \ldots, \widehat{A_k}, \ldots, A_n]\cdot f(A_k)\right);$$

here X_μ denotes the barycenter of μ, and $[A_0, A_1, \ldots, \widehat{A_k}, \ldots, A_n]$ denotes the subsimplex obtained by replacing A_k by X_μ; this is the sub-simplex opposite to A_k, when adding X_μ as a new vertex. Vol_n represents the Lebesgue measure in $\mathbf{R}^n$.

In the case of closed balls $K = \overline{B}_R(a)$ in $\mathbf{R}^3$, $Ext\,K$ coincides with the sphere $S_R(a)$ and the recent paper by Dragomir [2] illustrates the classical Choquet theory in the case where μ is the normalized Lebesgue measure on $\overline{B}_R(a)$:

$$f(a) \leqslant \frac{1}{\mathrm{Vol}\,\overline{B}_R(a)}\iiint_{\overline{B}_R(a)} f(x)\,dV \leqslant \frac{1}{\mathrm{Area}\,S_R(a)}\iint_{S_R(a)} f(x)\,dS.$$

His argument, based entirely on Calculus, avoids Choquet's theory, but it cannot be extended to arbitrary compact convex sets K and arbitrary Popoviciu measures on K.

A final remark concerns the case of non-metrizable compact convex sets K. As noticed E. Bishop and K. de Leeuw (Cf. [11], p. 7), in this case the set of extreme points of K need not be a Borel set. However, by combining the argument of Theorem 4 above with their approach in the case of probability measures (Cf. [11], p. 24) we obtain the following Choquet type theorem:

THEOREM 5. (*The generalization of the Choquet-Bishop-de Leeuw Theorem*). *Let μ be a Popoviciu measure on a compact convex subset K of a locally convex Hausdorff space E. Then there exists a probability Radon measure λ on K such that the following two conditions are verified*:

i) *$\lambda \succ \mu$ and λ and μ have the the same barycenter*;

ii) *λ vanishes on every Baire subset of K which is disjoint from the set of extreme points of K*.

Our final remark concerns the necessity of hypotheses in the right hand side inequality in (Ch). Precisely, it works beyond the framework of Popoviciu measures, an example being $(x^2 - x)\,dx$ on $[-1, 1]$. See Fink [3], p. 230.

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